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General Quantization[1]

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In the Darwinian evolution of physical theories, stability (genericity) has survival value. To convert a singular physical theory based on Lie algebras of several levels into a generic quantum theory with the same levels and nearly the same predictions and symmetries in a limited correspondence domain, it suffices to simplify the algebra of each level by a small homotopy (general quantization). This extends and unifies special relativization, general relativization, and canonical quantization. For exercise I general-quantize the scalar meson field in Minkowski space-time. The predictions of the resulting theory are finite, including its zero-point energy.

1. QUANTIZATION AS REGULARIZATION

Quantum theory began as ad hoc regularization prescriptions of Planck and Bohr, rigged up to handle some of the infinities that blocked earlier theories of the electromagnetic field and the nuclear atom. Then Heisenberg discovered that a slight change in algebra regularizes and at the same time improves agreement with experiment. We take regularization as the guiding goal for further quantization.

The paradigm is the linear harmonic oscillator of natural frequency ω. This is a continuous system in classical mechanics, where $pq - qp = 0$, but in quantum mechanics $pq - qp = -i\hbar$, and for any $\hbar > 0$, no matter how small, the oscillator is an aggregate of a variable finite number N of finite bosons of fixed energy $h\omega$ each, with total energy $N\hbar\omega$. Since N is unbounded the quantum theory is still singular, but less so.

For many the infinities that still haunt physics cry for further and deeper quantization, but until recently there has been little indication of exactly what and how to quantize. Quantization provides microstructure from the top down. In the absence of a more powerful quantization algorithm people have had to make daring hypotheses about microstructure from the bottom up, such as spin networks, strings, and loops. The top-down construction starts from a correspondence

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principle that connects theory and experiment, while bottom-up constructions start with a considerable gap between theory and experiment.

Canonical quantization not only regularized singularities but also continued a march toward group simplicity that special relativization began. Segal (1951) noted that our present Lie groups differ only infinitesimally from simple ones and proposed that this is the direction for further quantization. Vilela Mendes (1994) initiated the work in that direction and made considerable progress.

There are encouraging signs (Section 2) that when the Lie groups of the theory at last become simple, the theory becomes finite. We infer that our present infinities call not merely for further quantization but for quantization to the point of simplicity.

Each non-simplicity of the operational algebra in turn arises from an idol of the theory. We use the term idol in the sense of Bacon (1994), expecially his idols of the theater. Idols are false absolutes, constructs that change imperceptibly in ordinary experience and are therefore erroneously supposed to be fixed, able to act but not to react, like classical time or classical phase space. Idols couple into other constructs under invariance transformations and suffer no converse couplings. This suggests that quantum theories today are obstructed by idols erected by physicists of the past. Today it may be more practical to topple these idols than to continue to detour them.

Group theory provides a systematic way to detect and relativize some lethal absolutes. A theory has a group, and its absolutes have invariant subgroups that respect them and make their over-group compound (not semisimple). Idols correspond to ideals of the group or the Lie algebra. We relativize the absolute by simplifying the Lie algebra. This eliminates the invariant subgroups and the idol.

Moreover an arbitrarily small homotopy of the structure tensor suffices to simplify many Lie algebras. *Warping* (Section 3.1) a Lie algebra is the general process of applying a homotopy to its structural tensor, keeping the elements fixed, that makes the Lie algebra less commutative and closer to simple, for example diminishing the radical or the isotropic space of the Killing form. Warping is complete when the final Lie algebra is simple. Warping is a key step in special and general relativization, canonical quantization and general quantization.

Canonical quantization warps only the highest-level Lie algebra, and that not all the way to simplicity and finiteness. General quantization extrapolates canonical quantization in both respects. It warps the Lie algebras on all the known levels of a physical theory, and it warps them all the way to simplicity and finiteness. It does this by arbitrarily small changes in the structure tensor, so that it makes only small changes in experimental predictions for transformations not too far from the group identity, in the domain of correspondence between the warped and unwarped theories.

For exercise and illustration we general-quantize the scalar meson quantum field. A first-level quantization resolves the ether, the ambient medium, into a q series of many identical finite quantum elements, which are likely composite in turn, and so should be likened to crystal cells or molecules rather than atoms. A second-level quantization regularizes the field. The vacuum is the ambient mode of the ether, represented by a mode-vector $|0\rangle$. General quantization infers structures and symmetries for the ether and its elements from the structure and symmetry of the present-day vacuum by a routine heuristic procedure based on correspondence, simplicity, and symmetry.

Simple Lie algebras have quite special dimensions. There is no simple Lie algebra of dimension 2, for example. Therefore general quantization often requires us to introduce new dynamical variables into the theory, called regulators, to bring its dimensionality up to that of a simple group, before warping to simplicity. Then to freeze out these new variables and recover the singular theory, we must also hypothesize self-organization (crystallization, condensation, spontaneous symmetry-breaking). Special relativity and canonical quantum theory are exceptional in this regard. We guide ourselves through this phase transition as follows.

2. PHILOSOPHICAL BACKGROUND

The following philosophical remarks are included only to explain how general quantization was formed, not to somehow justify it. The theory must stand or fall on experiment, not on philosophy.

2.1. Less is Different too

More is different (Anderson, 1972); different from less, one understands. That is, when we pass from small to large numbers of systems we encounter spontaneous organization that increases structure and decreases symmetry, as in phase transitions like crystallization.

It follows that less is different too; different from more, of course. When we pass from large to small numbers of systems, and from the ether to the sub-ether, we expect to encounter a loss of organization, more symmetry, and less structure.

Discretization destroys continuous symmetry, quantization increases it. Bottom-up models of the sub-ether, like vortex, network, string, and loop models, enrich its structure, reduce its symmetry, and increase its singularity. General quantization leads only to conservative models like quantum theory and relativity theory, which increase symmetry and reduce singularity by breaking idols. To be sure, one could regularize a symmetry-decreasing model too by general quantization.

Early on Wigner noted that some important evolutions are small homotopies of the algebra. Segal suggested that these are in the direction of (group) simplicity (Segal, 1951). He explained this as an essentially Darwinian evolution, based on natural selection for stability (Section 3.1). After all, our experiments are disturbed by the many uncontrolled quantum variables of the experimenter and the medium. Our measurement of the structure tensor must err. To survive a physical theory should be stable against small errors in the structure tensor. The way out of the dark forest of instability is to always go downhill, that is, toward stability.

This criterion is based on dubious implicit assumptions about the domain of possibilities. For instance, groups that are stable in the domain of Lie groups are unstable within the larger domain of quantum groups or non-associative products. The group-stability criterion might produce some useful theories, but it might also exclude some. I note another criterion that can take its place.

There are encouraging signs that when the algebra is simple the theory is finite. Then infinities today result from departures from algebra simplicity, in turn caused by idols that must be relativized.

One can illustrate this regularization-by-relativization with the same elementary example as before (Section 1). The quantum linear harmonic oscillator has compound and singular Lie algebra and infinite-dimensional mode-vector space. Its basic coordinate and momentum operators diverge on most of its mode vectors. Segal stabilized this algebra by warping it to SO(3) (signature unspecified) which has an irreducible representation $R(l)$SO(3) of finite dimension $2l + 1$ for any finite quantum number $l = 0, 1, 2, \ldots$. For all our finite experiments can tell us, one of these matrix representations might works at least as well as the singular one. Yet its coordinate and momentum operators both have a discrete bounded spectrum with at most $2l + 1$ values, and are defined and finite on any vector, together with their products. This warping regularizes the theory as well as stabilizing it in some degree.

For another example where a well-chosen homotopy replaces infinite-dimensional representations of a compound group by finite-dimensional ones of a simple group (see Kim and Wigner, 1987, 1990).

In general, the irreducible representations of compound (= non-semisimple) algebras useful in physics are unique but contain serious infinities, while infinitesimally nearby simple algebras are non-unique but finite. It seems plasuble that some of these nearby finite-dimensional algebras suffice for present physics at least as well as the present infinite-dimensional ones.

2.2. Theory Drift

Words carry us fast and far, so it is important to take the right ones. Quantum mechanics use the concept of "state vector" and the quantum principle correctly despite the name but deeper thinkers have spent research lives trapped by this improper vestige of a wrong theory. The most careful formulation will not stop us for long from assuming that anything we call a "state vector" must describe the system. To fix the interpretation we avoid this term for the io (input–output) mode vectors or bras and kets. We use these vectors to represent not physical waves, nor observables of the system, but io actions by the experimenter that begin and end experiments. One never learns the vectors to put into the Malus–Born quantum principle by looking at the system, always by looking at the apparatus. They have no exact counterpart in the formal *c* theory, because *c* theories suppressed mention of the experimenter as a matter of philosophical principle. Schrödinger connected these vectors to Hamilton principal functions, which are flow generators, not states. The quantum system has no mode vectors, "state vectors," or whatever we call them. The experimenter has them. They are fully describable classical objects. The quantum system is not.

Then most of the universe is left out of any operational theory as a matter of principle and the concept of a final theory makes no operational sense. Then theory change is the norm and it behooves physicists to study it. Warping is one mode of theory drift that lends itself to mathematical study.

2.3. The Oldest Game in Town

Here are some notes on the history of theory-warping.

All the deep changes in the structure of successful physical theories since 1900—special and general relativity, quantum theory, gauge theory—have introduced warpings. Both relativity and canonical quantum theories have correspondence principles that imply a homotopy from the new theory to the old, without using the word homotopy.

Segal (1951) worked on the inverse problem, the one that concerns us here. How should present theories evolve into future ones? Wigner's first publication in the field, that with Inönü, dealt with the well-posed direct problem of how present theories contract to past ones. It specializes and inverts the homotopy of Inönü and Wigner (1953).

Space-time curvature and space-time quantization are dual warpings, of space-time and momentum-energy respectively. Canonical quantization and special relativization warp a classical Lie algebra to a quantum or relativistic one (Segal, 1951; Inönü and Wigner, 1953).

When the classical theory uses multiple quantification (higher order set theory), so does its general quantization. Weizsäcker proposed multiple quantification under the name of "multiple quantization," and the algebraic apparatus for multiple quantification exists (von Weizsäcker, 1955; Finkelstein, 1961, 1972a,b, 1974; Finkelstein *et al.*, 1974).

Vilela Mendes (1994) seems to have been the first to apply the stability principle to construct new quantum physics. He noted that to simplify most Lie algebras one must first introduce new variables and then invoke crystallization to freeze them out in the vacuum. He was apparently inspired by the mathematical theory of stable (= rigid) algebraic structures (Gerstenhhaber, 1964), which in turn may have been influenced by Segal's proposal.

People have since warped the stationary theory of a quantum harmonic oscillator (Kuzmich *et al.*, 1998, 1999, 2000; Carlen and Vilela Mendes, 2001; Atakishiyev *et al.*, 2003; 'tHooft, 2003; Baugh, 2004; Shiri-Grakani, 2004) and some its canonical dynamics (Baugh, 2004; Shiri-Garakani and Finkelstein, 2003). Madore's (Madore, 1992; Madore and Hitchin, 1999) "fuzzy spheres" include the Segal (1951) warping of the Heisenberg algebra $d\mathrm{H}(1)$ with one coordinate and one momentum as a special case.

The present concept of regularization by general quantization stems from the proposal of Segal (1951) for stabilization by simplification (section 3.1). It extends the stabilization of space-time by Vilela Mendes (1994) to higher levels, and seems to regularize as well.

2.4. Relativism

Physical theories, including the most relativistic, begin with absolutes. For example, special relativity renounces absolute time but keeps as absolute the class of all timelike directions. This class is then renounced by general relativity and replaced by a still higher type of absolute, the class of all metrical forms of Minkowskian signature.

These absolutes conflict with general relativism, the doctrine that *all* is relative, a philosophical position centuries older than general relativity. They also conflict with complementarity. We only allow entities into our theory that can be experienced, directly or indirectly. Experience is an interaction that changes both participants, in properties complementary to what is experienced.

If we hold both of the above views, we may infer that any physical theory is provisional, to be replaced when we study its assumed absolutes under sufficent resolution.

This suggests that physics ought to study the ongoing evolution of physics, and not leave it to historians. The search for a mathematical theory of small changes in physical theory took me back to the classic work on the small changes in physical theory sometimes misleadingly called "revolutions": Inönü and Wigner (1953). Inönü and Wigner do not suggest future warpings, nor consider the quantum story, but they point to Segal (1951), who had done both.

Canonical quantization has been so useful that it too has become an idol (in the Baconian sense). Many assume that the next theory, like the most recent ones, must have a canonical form. This idol creates instability, and therefore singularity, which can be eliminated by general quantization.

3. GENERAL QUANTIZATION

We (general-) quantize here singular theories that are based on some underlying Lie algebra $L(0)$ that is not simple, and on a representation thereof—call it $R(0)L(0)$—that is not regular. In some important singular theories the representation $R(0)$ is uniquely determined by unitarity, irreducibility, and one quantum constant $\hbar$.

By a *central invariant* we mean an algebraic combination of Lie algebra elements that is central in every representation, and therefore *ac* number in every irreducible representation. For any Lie algebra L and any representation R of L, the Casimir invariant C_n of R is the coefficient of z^n in the invariant characteristic polynomial

$$C(z) = \det(L - z\mathbf{1}) = \sum C_n z^n \tag{1}$$

for $L \in RL$ and is a central invariant. Planck's constant in the form $i\,\hbar$ is the value of the central invariant $r = i\,\hbar$ for $d\mathrm{H}(1)$.

It is convenient to introduce dimensional constants δq_n to bring the generators q_n to a standard dimensionless form L_n whose spectrum has unit spacing. Then the C_n have integer eigenvalues c_n. The *quantum numbers* c_n define a representation algebra $R(\mathbf{c})L$.

The algebra does not define the physics. One must describe its elements physically to give it physical meaning. If we double the value of $\hbar$ we change the physical predictions greatly but do not change the algebra, up to isomorphism. We take the physical variables q_n as a distinguished *physical basis* $\mathbf{B}$ within the operator algebra L, defined by how we measure them in standard units: for example, position, momentum, ... The *quantum constants* $\mathbf{h} = \{\delta q_n\}$, including $\hbar$, then define the representation $\mathbf{B}(\mathbf{h}, \mathbf{c})$ of the preferred basis.

To quantize a singular theory in the present general sense we:

1. Warp its Lie algebras to simple ones with as few new variables as possible.
2. Choose representations that correspond with the singular theory in the experimental domain.

In the first step we make a finite choice among several possible simple Lie algebras. In the second step we guess or measure some discrete quantum numbers and some continuous quantum constants. The correspondence principle provides experimental meanings for some of the variables.

Often the singular theory uses several singular algebras. For example, classical mechanics has both a commutative algebra of phase-space coordinates and a Lie algebra of phase space coordinates with the Poisson Bracket as product. Classical space-time has a commutative algebra of coordinates and a Lie algebra of vector fields with the Lie Bracket as product. In such cases economy prefers a quantization that deduces both singular algebras as singular limiting cases of one more regular algebra, as did canonical quantization.

3.1. Warping

Some terms: For any vector space V, $\mathrm{D}V = V^{\mathrm{D}}$ is the dual space to V. A † space V is a vector space over $\mathbb{C}$ provided with an involutory antilinear anti-automorphism $\dagger : V \to V^{\mathrm{D}}$. Vector automorphisms respecting † are called †-unitary.

A † is equivalently a Hermitian sesquilinear form $\dagger \in V^H \otimes V^{\mathrm{D}}$, not necessarily positive definite. In a quantum theory, unit vectors $\psi \in V$, $\psi^\dagger(\psi) = 1$, represent input modes; unit dual vectors $\phi \in V^{\mathrm{D}}$ express output modes; the transition amplitude is $A = \phi(\psi) = \langle\phi|\psi\rangle$, which is 1 (assured transition) when $\phi = \psi^\dagger$; and the † represents total time reversal (Finkelstein, 1996; Saller, 2006).

We deal with both abstract and operational algebras or groups. An operational algebra is an abstract algebra with a representation and an operational interpretation. It consists of linear operators on a † space. The interpretation may be expressed by assigning names like momentum or charge to its elements, which define how they are executed in the laboratory. It suffices to do this for a basis. Then an operational algebra is an abstract one provided with a representation and a distinguished basis. Similarly for groups. Stretching the preferred basis elements does not change the abstract algebra or the representation algebra but it changes the physical interpretation, and therefore the operational algebra.

An algebra is defined by a vector space A and a structure tensor $\times \in A \otimes A \otimes A^{\mathrm{D}}$ obeying well-known linear and associative conditions. A † algebra A is an algebra A provided with an involutory anti-automorphism $\dagger : A \to A$. For a † Lie algebra, $[a, b]^\dagger = [b^\dagger, a^\dagger] = -[a^\dagger, b^\dagger]$. The operational algebra A of a canonical quantum theory (also called the "algebra of observables," though the observables form a subset of measure 0 in A) has besides the operations $\times$ and †, a canonical imaginary $i = -i^\dagger = -1/i \in \mathbb{C}$ relating anti-Hermitian automorphism generators $a \in A$ to Hermitian observables $o = o^\dagger = i\,\hbar a \in A$, and changing sign under both total time reversal † and Wigner time reversal T.

An ideal I of an operational Lie algebra L is a subalgebra that obeys Lie algebraic relations $[L, I] \subset I$. These tell us that I may change some elements a in L, but does not change in reaction. Nothing in L changes I into something else. Thus an ideal in an operational algebra defines an idol.

A *stable* † Lie algebra is one whose Lie product $\mathsf{X} : a \otimes b \to [a, b]$ is isomorphic to all the Lie products $\mathsf{X}' \in N(\mathsf{X})$, a neighborhood of X, that are compatible with the same †. (Segal's (1951) discussion of stability ignores †.) Because the † is stable, we need not warp it.

About nomenclature: Some concepts have been formulated and named several times. Warping converts an algebra to one that is variously and synonymously said to be robust, rigid, stable, regular, or generic relative to the original one. Conversely the original algebra is said to be fragile, elastic, unstable, singular, or special. A homotopy that leads away from a singular limit was called a *deformation*. This now turns out to be backwards. It calls the less stable theory (say Galileo's) better formed than the stabler one (say Einstein's) and institutionalizes the preference for the singular and unstable that put us in our present fix. It is a vestige of the c idols that created the infinity problem in the first place. It is more heuristic to credit a homotopy with reforming the singular theory than deforming it. In the abstract theory the word "rigid" is customary, but the operational algebras in a neighborhood are slightly different in their interpretations, assuming that the distinguished basis is unchanged, and so we prefer the original term "stable" of Segal.

Semi-simple Lie algebras are stable (Segal, 1951) too. We can usually ignore the difference between the simple and semi-simple here. A direct sum is an incoherent mixture, and we see only one system at a time. One well-chosen maximal measurement will reduce a semi-simple operational algebra to one of its simple "superselected" terms for all subsequent measurements.

In elementary quantum theory the variables of a quantum system S are represented by operators on a mode-vector † space $V = V_S$ for the system. One always graduates to a space V that is also an algebra, whose multiplication composes the constituent units of the system; for example, the algebra of skew-symmetric tensors for fermions and symmetric for bosons. We assume from the start that the physical V is not merely a mode-vector space but a mode algebra. The operational algebra of the system is then an algebra over an algebra. It acts on the mode algebra through linear transformations, not necessarily algebra morphisms. Consistency requires that both the operational and the mode algebras should be stable.

A homotopy $A_0 \hookrightarrow A_1$, from one algebra A_0 to another A_1 (possibly Lie) on the same vector space A, each with its own product X_0 and X_1, is a continuous function $\mathsf{X} : A \otimes A \times I \to A$, where $I = [s_0, s_1] \subset \mathbb{R}$ is an interval, such that $\mathsf{X}(a, a', s_0) = a\mathsf{X}_0 a'$, $\mathsf{X}(a, a', s_1) = a\mathsf{X}_1 a'$, and $\mathsf{X}(a, a', s) = a\mathsf{X}_s a'$ is an algebra product for all $s \in I$. Usually $s_0 = 0$.

Segal uses the concept of a homotopy $A \hookrightarrow A(s)$ from an unstable algebra $A = A(0)$ to more regular, more stable algebras A_s (say, with smaller nilradicals) for homotopy parameter $s \in (0, s_1]$, without naming the concept. Since it increases non-commutativity, a generalized curvature, we call such a homotopy a *warping*.

As an example Segal warps a canonical Lie algebra of q, p, i to a Lie algebra of three generating angular-momentum-like variables $(\widehat{q}, \widehat{p}, \widehat{r})$, replacing the central i with the non-central r. His homotopy transformation depends quadratically on the homotopy parameter. We call the inverse homotopy $A(s_1) \hookrightarrow A(0)$ of a warping, a contraction. What Inönü and Wigner (soon after) called a contraction is a is a special case that we call a linear contraction.

Linear contractions sufficed to contract special relativity to Galilean relativity and quantum theory to classical mechanics. To regularize canonical quantum theory requires a quadratic contraction; linear will not do. So do the regularizations of bosonic statistics and of space-time structure. The inverses of these quantizations are all contractions in the more general non-linear sense.

Like Segal, we are mostly concerned with the inverse problem, warping present theories to future ones. The direct problem, contracting present theories to past, is of historical interest, and it provides our precedents. Stabilization is an inverse problem: returning from the singular limit to the regular case. Like many inverse problems, it is badly posed and has no unique solution.

In matrix representations of a † Lie algebra, we require that the † be represented on the matrices by Hermitian conjugation, possibly with an indefinite metric $\dagger_{\hbar m}$. We may require warping to conserve the † without loss.

We have guides for each step. We add only enough variables to make the algebra homotopic to a simple one. We choose the warped Lie algebra close to the unwarped. We choose the quantum numbers so that the regular and singular theories agree as closely as necessary in the experimental domain.

Classical predicates are binary-valued variables, taking values 0 (false) and 1 (true). Classical predicates commute, Boole noted, but quantum predicates do not, according to Heisenberg. Quantum logic is non-commutative logic. Von Neumann's non-distributive logic is substantially equivalent but unwieldy. To consolidate quantum theory and relativity requires us to replace classical logic with quantum logic throughout, especially in space-time geometry. Previously we attempted to do this from the bottom up (Finkelstein, 1961, 1972a,b, 1974) with minimal success. Here we work from the bottom down with more success, actually producing a theory. Instead of guessing at the chronon, the atomic unit of the net, we construct it from the surface structure by general quantization. We do not guess at constituents of nature; quantization provides what it needs for finiteness, up to a small number of discrete choices and parameters.

Because the road has been so long, one would think that regular theories are something rare and special, lone diamonds hidden in much clay; and that divergence is the norm. On the contrary, obviously it is singularity that is singular. Regularity is the generic case. One must fixate on assumptions of probability 0—the remaining idols of classical physics—to make a theory singular. General quantization softens those assumptions. Our main work here is to general-quantize space-time, statistics, and dynamics.

3.2. Relativized Space-Time

The general quantization of space-time relativizes the space-time event. Working quantum theories today start from a Lagrangian density. This concept, independent of the details of any particular Lagrangian, is built on a non-experimental idealization of space-time events that make its algebra singular both in the small and in the large. Canonical quantization converts a c Lagrangian into a less singular theory based on a Feynman amplitude for a c history, but a theory which is not yet regular. This theory can still be regarded as a picture in space-time (in Feynman's term) and it is therefore still singular, though less so than the c theory. Further warping then converts the Feynman amplitude into a mode vector for a quantum history. The result is no longer a c space-time picture and is no longer singular.

Warping space-time eliminates real space-time points, c or q, in the sense that special relativity eliminated real points of time. The infinitesimal space-time diffeomorphisms in the Einstein Lie algebra couple x^{μ} into ∂_{μ} but not conversely. This is how the Einstein Lie algebra is compound. The concepts of space-time point and therefore of scalar field $\phi(x)$ are idols of general relativity, and any warping that simplifies the Einstein Lie algebra must break them.

Bergmann noted that Dirac's historic quantization program for gravity had eliminated absolute space-time points from the quantum theory of gravity. He said that the world point itself possesses no physical reality (Bergmann and Komar, 1972; Bergmann, 1979), in the same sense that Minkowski said that space and time points possess none. There is a simpler road to this conclusion. Clearly space points are abstractions from small material classical bodies, and space-time points from events in the history of these bodies. Since at the microscopic level there are no such bodies, there is no reason to suppose that there are such points. Since physical events are actually composed of quantum processes, presumably physical space-time points are actually composed of similar quantum entities.

The theory of Vilela Mendes and the development represented here are not built on classical space-time (ST) points. General quantization analyzes space-time into elementary q transition processes, represented in a stable algebra that fuses and unifies space, time, the imaginary i, momentum, and energy (STiME). This greater unity distinguishes the space structure of Vilela Mendes (1994, 2005) and the present work, based on the simplicity doctrine, from the quantum spaces of the "space-time code" (Finkelstein, 1961, 1972a,b, 1974, 1996), which did not use algebra simplification and stability.

The space-time continuum is not a fundamental structure but arises from STiME in a singular limit of an organized mode of an underlying complex system. To avoid seeming oxymorons like "organization of the vacuum" we call the underlying system the net and its ambient organized mode the ether, with the understanding that the ether determines no rest frame. STiME splits into the usual fragments—space-time, the complex plane, and momentum-energy—only relative to the ether.

The net supports a basic kinematic symmetry between space-time and energy-momentum variables like that postulated by Born *et al.* (1949) and Born (1949) and co-workers in their reciprocity theory, except that now it extends to i as well. The ether condensation breaks this symmetry.

3.3. Quantum Constants

The quantization of Minkowski space-time exhibited here has chronons with warped bosonic statistics and the symmetry group SO(5, 1). It is a transient theory and should not be regarded as final but some of its features indicate what to expect. For one thing, it is intrinsically non-local in both space and momentum variables with respective non-localities δx and δp. It also has an invariant integer parameter N, a maximum number of elementary processes. The ether crystallization breaks Born reciprocity in the singular limit $\delta x \to 0$, $\delta p \to \infty$, $\mathsf{N} \to \infty$, and makes the singular limit theory local in space-time but not in energy-momentum. That is, in a single interaction there is no finite change in position or time, but an arbitrarily large change in momentum and energy; the standard assumption.

In general the regulation process introduces new regulation operators or *regulators* q_n and four kinds of physical constant with relations among them:

1. Signatures defining the Lie product operation $\widehat{\mathsf{X}}$.
2. Regulation constants or *regulants* $\overline{q}_n$, expectation values in the ambient ether.
3. *Quantum numbers* $\mathbf{c}$ defining a representation $R(\mathbf{c}) : L \to A(\mathbf{c})$ of the Lie algebra.
4. *Quantum constants* δq_n defining the representations of selected physical operators within $A(\mathbf{c})$.

The regulants $\overline{q}_n$ are typically both spectral maxima and ambient values of regulators $|q_n|$,

$$\max |q_n| = \langle 0|q_n|0\rangle := \overline{q}_n. \tag{2}$$

If q has a uniformly spaced spectrum we designate the spectral spacing, the fixed quantum of q, by δq.

We warp the canonical relation $pq - qp = -i\,\hbar$ to the cyclic form

$$\widehat{p}\widehat{q} - \widehat{q}\widehat{p} = \frac{\delta p\,\delta q}{\delta r}\widehat{r}, \quad \&\text{ cyc} \tag{3}$$

on dimensional grounds. The operator that freezes to $-i\,\hbar$ in the singular theory is clearly

$$\widehat{\imath\hbar} = \frac{\delta p\,\delta q}{\delta r}\widehat{r} := N\delta p\,\delta q. \tag{4}$$

where the integer N is the maximum eigenvalue of $|\widehat{r}|$ as a multiple of its quantum δr.

Canonical quantization and special-relativization introduced scale or quantum constants but no regulators. Subsequent warpings have both (Segal, 1951; Vilela Mendes, 1994; Shiri-Garakani and Finkelstein, 2003; Baugh, 2004; Shiri-Grakani, 2004).

3.4. Non-Uniqueness

The simplicity principle provides the kind of over-all understanding of the development of physics that Darwin's theory of evolution and Wegener's theory of continental drift supply for biology and geology. It does not determine the development but suggests several possibilities for experiment to choose among.

General quantization produces a phenomenological theory, not what can be called a "fundamental theory." Its dynamics is not an absolute law but a partial description of the action of an otherwise ignored background. Since it resolves more singularities, and describes more excitations of the net than gravitational alone, it generally introduces physical constants besides Planck's quantum of action and light-speed, to be determined empirically. It also leaves open a discrete choice between the orthogonal and the unitary line of algebras, and discrete choices of signature, that must also be decided empirically. But it produces finite physical theories that were inaccessible before.

In some singular cases, like the harmonic oscillator, the Lie algebra uniquely singles out an irreducible representation by an associative operator algebra. Warping L leads us to several candidate simple Lie algebras $\widehat{L(\sigma)}$ near L, distinguished from each other by signatures σ, for example. Each of the $\widehat{L(\sigma)}$ in turn has many candidate irreducible associative unitary representations $R(\mathbf{c})\widehat{L}(\sigma)$ distinguished by a sequence $\mathbf{c}$ of invariant quantum numbers. In the singular limit, some of the $\mathbf{c}$ go to ∞. Finally one must choose quantum constants $\mathbf{h}$ to specify a physical basis $B(\mathbf{h})$ Experiment must determine the best values of the quantum constants and numbers.

3.5. Stages of General Quantization

We classify theories as c or q as their dynamical variables all commute or not and divide q theories into q/c and q/q as their time is commutative or not. We formulate a q/q physics here, but the working physics of today is still q/c, and some of the current intuition is still c. For clarity I distinguish the three cases explicitly before setting to work.

3.5.1 Theories c

The c view of nature assumes that the universe, and every isolated system in it, has a complete numerical description or state (q, p) that assigns values to all its variables and determines everything that it does. The c can stand for commuting and central as well as classical. The experimenter is unmentionable for polite c theories; Bell criticizes q theory severely for violating this code.

Two ways to formulate a c dynamical theory have q correspondents. The Hamiltonian formulation is synchronic, single-time, assembles the operational algebra from instant algebras; the Lagrangian, diachronic, many-time, carves the operational algebra out of a larger algebra of kinematically possible histories. We use the diachronic.

3.5.2 Theories q/c

The q/c physics of Heisenberg, Bohr, and the Standard Model is incomplete as a matter of principle, in the sense that it fails to answer most well-formed questions about experiment; but less so than the c view, since it includes the experiment and acknowledges its incompleteness explicitly in the Malus–Born quantum principle

$$A = \phi^{\dagger}\psi. \tag{5}$$

A is the transition probability amplitude from the sharp experimental input mode or channel ψ to the output one $\phi^{\dagger}$ for the system. This is invariant under the unitary group; and also under total time reversal, a dual symmetry $\dagger$ between these channels, that exchanges io (= input–output) channels ψ, $\psi^{\dagger}$ and complex-conjugates A. In the dynamical equation for any variable x

$$\frac{dx}{dt} = \frac{1}{i\hbar}[H, x] + \frac{\partial x}{\partial t} \tag{6}$$

the variable t commutes with all measurable quantities of a q/c theory.

Canonical quantization replaces basic c variables q, p by non-commutative quantizations $\widehat{q}$, $\widehat{p}$. When the c state of the c system is the variable pair (q, p), by continuity or correspondence the q/c state of the q/c system is the operator pair $(\widehat{q}, \widehat{p})$. Since the two variables do not commute the q/c state is not observable.

The synchronic and diachronic c formulations have q/c correspondents. The relativistic action principle is diachronic. The action, mysteriously non-operational in the c theory, now becomes the phase of the history mode-vector, according to Dirac, which can be approached experimentally by observing quantum interference patterns.

When the synchronic theory is singular, the diachronic theory is even more singular, because the system has many more history modes than single-time modes. If the space-time and the synchronic theory are regular, however the diachronic theory will likely be too.

3.5.3 Theories q/q

In a q/q theory, the algebras of all levels within the theory are non-commutative. To regularize such hierarchic theories we must regularize all their constituent algebras and the algebraic relations between levels. For this we use an algebraic concept of quantification (Section 4) or statistics.

3.6. Three Lines of Theory

Shall we follow the orthogonal, unitary, or symplectic line of simple algebras? We work with huge dimensionality, so the exceptional algebras do not come in. Experiment does not yet clearly decide our choice.

Here I take the algebra requiring the fewest regulators, whatever its line. Baugh (2004) takes the A line. Baugh may well prove to be right, but life seems easier along the D line than the A or C. For example: The Heisenberg algebra $d\mathrm{H}(3, 1)$ of Minkowski space-time has 9 dimensions. Its orthogonal simplification is $d\mathrm{SO}(6)$, with 15 dimensions, requiring $15 - 9 = 6$ regulators Vilela Mendes (1994), while its unitary simplification is $d\mathrm{SU}(6)$, with 35 dimensions, requiring $35 - 9 = 26$ regulators. By Ockham's principle I choose the D line for these initial studies.

The principle that less is different, however, suggests the A line. The Heisenberg algebra $d\mathrm{H}(N)$ is invariant under $\mathrm{GL}(N, \mathbb{R})$ of dimensionality N^2. If less organization means more symmetry, the warping $d\widehat{H}(N)$ should have at least the symmetry group $\mathrm{GL}(N, \mathbb{R})$. The orthogonal warping of $d\mathrm{H}(N)$ is $d\mathrm{SO}(N + 2)$, which has the symmetry algebra $d\mathrm{O}(N + 2)$ of dimensionality only $(N + 2)(N + 1)/2$. If $N4$ then $(N + 2)(N + 1)/2 < N^2$, and the A line is indicated. For $N = 4$ there is the famous coincidence $d\mathrm{SO}(3, 3) = d\mathrm{SL}(4)$ and the A and D lines seem equally open.

I mention another sign that inclines one toward the A line. The group of the regularized theory includes all the stable groups of the singular theory. The low-dimensional groups of present physics like the Lorentz $\mathrm{SL}(2, \mathbb{C})$ or isospin $\mathrm{SU}(2)$ straddle both the A and D lines. The first group that does not evade the question is color $\mathrm{SU}(3)$, and it is on the A line.

4. QUANTIFICATION

The passage from a one-system theory to a many-system theory is a general process aptly named quantification by the Scottish logician William Hamilton (1788–1856). It is not a quantization but something much older.

The operations of system creation and annihilation go on off-stage in the one-quantum theory and are represented by mode vectors, not operators. Quantification brings them on-stage and represents them by operators, not mode vectors. The notion that experiments on a single quantum can tell us the operational algebra of a many-quantum system is a relic of c physics, where bodies are made of atoms and the body state space is the Cartesian product of many atom state spaces, but it has worked amazingly in the q theory, with necessary changes.

The earliest quantum statistics naively took for granted that the many-quantum mode space is the tensor algebra over V, $\mathrm{Tensor}V$. This is Maxwell–Boltzmann statistics, the quantification for fictitious quanta that we can call maxwellons. Then it was realized that the io operators generated the quantified algebra must obey significant commutation relations. For example the bosonic and fermionic quantifications Σ_σ are based on the algebraic relation

$$b^\dagger a = \sigma a b^\dagger + \langle b|a\rangle \tag{7}$$

for $a, b \in V$. We confine ourselves here to $\sigma = +$ (bosonic), $\sigma = -$ (fermionic), $\sigma = 0$ (maxwellonic) and their regularizations and iterations.

The Lie algebra (7) for $\sigma = +$ is singular. We generalize a bit to permit regular statistics. By a quantification Σ I mean a construction that sends each one-quantum operational algebra A_1 to a many-quantum operational algebra $A_\Sigma V \supset A_1$ with A_1 as a subalgebra. And it does this by giving the io mode space $W := V \oplus V^{\mathrm{D}}$ of A_1 the structure X of a Lie algebra that is represented in A_Σ by an irreducible †-unitary representation. Now this Lie algebra can be regular.

One must specify a many-quantum vacuum to define a unique many-quantum input mode-vector space as well. In c thought and in boson statistics the vacuum is absolute. We suppose that this is a singular limit and relativize the vacuum projection $P(\mathrm{vac}) \in \Sigma V$. A vacuum $P(\mathrm{vac})$ defines a mode-vector space $\Sigma V\ P(\mathrm{vac})$. If we bring $P(\mathrm{vac})$ to diagonal form, all the operator products in $\Sigma V\ P(\mathrm{vac})$ are matrices with only one non-zero column, which represents a ket.

To define a quantification we must give not only the Lie algebra but must also give values $\mathbf{c} = \{c_n\}$ for its central (Casimir) invariants $\mathbf{C}$ to defines an irreducible representation $R(\mathbf{c}) : L \to A$ in a representation algebra $A(\mathbf{c})$ (the endomorphism algebra of a † space). Then up to isomorphism we define

$$\Sigma(\mathbf{c})A_1 := A(\mathbf{c}). \tag{8}$$

In the bosonic case, $\mathbf{c}$ contains only Planck's constant. In the fermionic case, canonical anti-commutation relations define a graded Lie algebra; we will leave the grade implicit when we speak of Lie algebras.

We can regard the representation algebra $\Sigma(\mathbf{c})L$ for any Lie algebra L as a generalized statistics or quantification for quanta that we can call "L-ons." Bosons and fermions result from canonical and Clifford (graded Lie) algebras respectively. One must warp these algebras to regularity too. We already unwittingly regularized the fermionic statistics for reasons of quantum logic (Finkelstein, 1982; Wilczek, 1982). We regularize the bosonic statistics in Section 6.3.

Thus the basic algebras of quantum physics, such as the Heisenberg Lie algebra with $pq - qp = i$, and the Fermi graded Lie algebra with $pq + qp = 1$, admit two interpretations: Geometrical, as symmetries and variables of a classical continuum, as when p represents an infinitesimal translation along the q axis. And logical or statistical, defining a quantification. Since the classical continuum is singular, we regard all our Lie algebras as ultimately statistical.

Dynamics has a hierarchy of at least five algebras (6). In formal logic such hierarchies are handled with quantifiers. In q/c physics the lower level c

quantification is handled informally and intuitively, and the higher q level quantification is constructed from the lower algebraically as in Section 4. In q/q physics we must handle all quantifications algebraically.

Quantification deceptively resembles quantization in more than spelling. Both adduce commutation relations, and they may even end up with the same algebra. Nevertheless they are conceptual opposites and if they come to the same place, they arrive there from opposite sides. Quantification sets out from a one-quantum theory. Quantization set out from a classical theory, which is a many-quantum system seen under low resolution and with many degrees of freedom frozen out. For extremely linear systems like Maxwell's, the two starting points may have similar-looking variables but the operational meanings of those variables are as different as c and q.

5. REGULARITY AND STABILITY

Simple algebras are stable (Segal, 1951; Gerstenhhaber, 1964; Vilela Mendes, 1994). So are semi-simple ones, but these are direct sums of simple ones, and in quantum theory a single well-chosen measurement reduces a semi-simple algebra to one of its simple terms, so the difference is not crucial. In what follows we implicitly leave the possibility of semi-simplicity open.

Simple Lie algebras seem to result in finite (= convergent) theories. We begin to explore this delicate question here. Certainly simple Lie algebras have complete sets of finite-dimensional representations supporting finite-dimensional quantum theories with no room for infinities. The simple algebras with indefinite metric have problematic infinite-dimensional irreducible unitary representations besides the good finite-dimensional ones. We hypothesize that we can approximate the older unstable compound theory without these infinite-dimensional representations; this has been the case for the Lorentz group, for example. If so, then simplicity pays in finiteness as well as stability. Then the division between stable and unstable algebras divides finite theories from infinite as well.

It also divides the mechanical theories with singular Hessian determinants from those with regular Hessians. Indeed, all singularities that depend on some variable determinant miraculously vanishing are non-robust, non-generic, unstable by that fact, and are eliminated by general quantization.

5.1. Stabilization by Warping

The Lie algebraic products $\mathsf{X} : V \otimes V \to V$ admitted by a given vector space V, also called structure tensors, form a quadratic submanifold $\{\mathsf{X}\}$ in the linear space of tensors over V, defined by the Lie identities

$$\mathsf{X}(a \otimes b + b \otimes a) = 0, \quad \mathsf{XX}(a \otimes b \otimes c + c \otimes a \otimes b + b \otimes c \otimes a) = 0. \tag{9}$$

The equivalence classes modulo Lie-algebra isomorphism cover the quadratic manifold {X} disjointly. Any singular Lie algebra lies on the lower-dimensional boundary in {X} of a finite number of these classes. For example, the 6-dimensional Galilean algebra of rotations and boosts sits between the SO(4) algebras and the SO(3, 1) algebras. To regularize such a singular algebra we merely move its structure tensor off this boundary to an adjacent simple algebra (Segal, 1951; Gerstenhhaber, 1964; Vilela Mendes, 1994; Saller, 2006). This is the core of the warping process, but it has ramifications extending through the whole physical theory. The warped group approximates the unwarped one only near their common point of tangency, as a sphere approximates a tangent plane. Part of the warping process consists of limiting the domain of the unwarped theory to this neighborhood, whose size is set by a physical constant or constants new to the singular theory, and which must include the experiments that have been satisfactorily described by the singular theory.

Unlike some forms of regularization process, such as discretization, warping never breaks a symmetry algebra but merely warps it slightly, and this always results in a unification of previously unrelated concepts. A warped theory $\widehat{\Theta}$, by fitting its regulants into the error bars of the unwarped theory Θ, inherits the operational semantics and past experimental validations of Θ, while still making radically new theoretical predictions about future experiments,

5.2. Regulators

The operator i is central, but not its warping $\widehat{\imath}$, which couples p and q, E and t. This unification of time and energy is even more bizarre than the unification of time and space, but this does not mean it is right. Likewise, warped regulators couple and thus unify operators that were uncoupled before warping.

If we introduce regulators we also need to explain how the unregulated singular theory could work as well as it does without them. Call the subspace of the regular mode-vector space where the regular theory agrees with the singular theory within experimental error, the *correspondence domain*. We hypothesize that self-organization freezes out the regulators in the correspondence domain, where the singular theory gives some good results. General quantization generally exposes a much larger symmetry algebra, supposed to have been hidden in the past by self-organization, and able to manifest itself in the future under extreme conditions like ether melt-down. Carried far enough, general quantization converts a singular theory with a compound algebra (= non-semisimple algebra) into a regular theory with a simple algebra (Segal, 1951). This requires no change in the stable elements of a theory, only in the unstable elements, such as the classical theory of space-time.

Suppose that the simple Lie algebra is an orthogonal one $d\mathrm{SO}(N)$ (rather than unitary or symplectic). Then we can choose each warped generating variable

q to be a multiple of an appropriate dimensionless component L^α_β of an angular momentum in N dimensions, by a dimensional constant δq:

$$\widehat{q} = \delta q L^\alpha_\beta. \tag{10}$$

We adjust the spectral spacing of L^α_β to 1. Then the quantum of q is δq. To diagonalize an antisymmetric generator L^α_β requires adjoining a central i for the purpose. Then the generators are all quantized with uniformly spaced, bounded, discrete spectra. The maximum of the absolute values of the eigenvalues of $\widehat{q_n}$ we designate by Max$\widehat{q_n}$.

These δ's generalize the quantum of action, $\delta A = \delta(E/\omega) = \hbar$, so we call them quanta of their variables. For example, warping introduces quanta δx of position, δt of time, δp of momentum, and δE of energy, as well as the familiar quanta of charge and angular momentum.

The main singular algebra of q/c physics, the Heisenberg algebra $d\mathrm{H}(M)$ (for M spatial dimensions), whose radical includes $i\,\hbar$, has already been warped for $M = 1$ (Segal, 1951; Vilela Mendes, 1994; Kuzmich *et al.*, 1998, 1999, 2000; Carlen and Vilela Mendes, 2001; Atakishiyev *et al.*, 2003; 'tHooft, 2003; Baugh, 2004; Shiri-Grakani, 2004; Czachor and Wilczewski, 2005) and for $M > 1$, both unitarily (Baugh, 2004) and orthogonally (Shiri-Grakani, 2004; Vilela Mendes, 2005).

6. A REGULAR RELATIVISTIC DYNAMICS

It remains to be seen whether the infinite-dimensional representations of the non-compact groups like the Poincaré group that are used in quantum physics today can indeed be approximated by a finite-dimensional algebraic representation of an approximating orthogonal group. In the non-compact cases the orthogonal groups have infinite-dimensional irreducible unitary representations as well as finite-dimensional orthogonal ones. The danger is that an infinite-dimensional representation is required for this approximation, with its native divergences.

A typical example: Consider a scalar quantum of mass m in a space-time of $3 + 1$ dimensions. One can approximate its singular Poincaré Lie algebra $d\mathrm{ISO}(3, 1)$ with a regular de Sitter Lie algebra $d\mathrm{SO}(5, 1) \to d\mathrm{ISO}(3,1)$. A scalar massive quantum in Minkowski space-time provides an infinite-dimensional unitary representation $R\, d\mathrm{ISO}(3, 1)$ in use today. Can one approximate this useful infinite-dimensional representation of the singular algebra by a finite-dimensional representation of the regular algebra?

The mathematical meaning of a singular theory is not well defined. A singular theory is not so much a theory as a dare: "Make a theory out of this if you can!" We do this here by warping the algebras of the theory, which also slightly changes its finite parts.

At least five major Lie algebras arise in such models:

Level	Space		Lie algebra
1	Space-time tangent space	$d\mathcal{X} = \{dx\}$	$d\mathrm{SO}(3, 1)$
2	Space-time	$\mathcal{X} = \{x\}$	$L_{\mathcal{X}}$
3	Field-value tangent space	$d\mathcal{F} = \{df\}$	$L_{d\mathcal{F}}$
4	Field-value space	$\Phi = \{\phi\}$	L_{Φ}
5	Field history space	$\mathcal{F} = \{f\}$	$L_{\mathcal{F}}$

The initial hierarchic structure is a lambda we assume, with space-time and field variable on the same level, and the final structure is simpler:

$$\begin{tikzcd} & & \mathcal{F} & & \\ & \mathcal{X} \arrow[ur] & & \Phi \arrow[ul] & \\ d\mathcal{X} \arrow[ur] & & & & d\Phi \arrow[ul] \end{tikzcd} \quad \rightarrow \quad \begin{tikzcd} \widehat{\mathcal{F}} \\ \widehat{\mathcal{X}} \arrow[u] \\ d\widehat{\mathcal{X}} \arrow[u] \end{tikzcd}$$

For the Lorentz group $L_{d\mathcal{X}}$ is regular and for the scalar field L_{Φ} and $L_{d\mathcal{F}}$ are the commutative Lie algebra on $\mathbb{R}$, also regular. We regularize the remaining algebras here.

6.1. Regular Space-Time

We regularize space-time first, then the scalar field on that space-time. This is mainly an illustrative example chosen as training for the most interesting singularity, that of gauge theory. The general quantization of gravity, in progress, suggests a different quantum space-time that we take more seriously.

The usual space-time coordinates x^{μ} commute and generate a compound commutative four-dimensional Lie algebra. There is no 4-dimensional simple Lie algebra. To make simplification possible without losing Lorentz invariance we adjoin the four differential operators ∂_{μ} and 1 as regulators, resulting in the compound Lie algebra $\mathrm{H}(4) = \mathrm{Lie}(x^{\mu}, \partial_{\mu}, 1)$ with standard commutation relations understood. This may also be the most economical way.

Now the irreducible unitary representation is essentially unique: The generators x^{μ}, ∂_{μ}, 1 act in the standard way on $\mathrm{L}^2(\mathcal{M}^4)$. This is also isomorphic to the diachronic pre-dynamical operational Lie algebra of a single scalar quantum particle in space-time. Statements about position in the abstract have been imbedded in statements about a quantum particle of unspecified dynamics, which we call "the probe." Inevitably this brings in statements about the momentum of the probe as well. This is but a partial regularization of space-time, neither regular nor simple.

Lie(x^μ, ∂_μ, 1) is also the Lie algebra $\Sigma_+ V(3, 1)$ of a certain bosonic aggregate. The mode-space $V(3, 1)$ of the individual boson is isomorphic to the tangent space $d\mathcal{M}^4$ to four-dimensional Minkowski space $\mathcal{M}^4$ at the origin but is not that space, being interpreted in a way that is non-standard for differential geometry. Its vectors are mode-vectors of a hypothetical quantum; the "minkowskion," let us call it. The classical space-time is now presented, ready for regularization, as a bosonic aggregate of minkowskions which has been reduced to a classical system by freezing out the momentum-energy variables, and centralizing ("superselecting") the coordinates x^μ, effectively restricting frames to the classical space-time coordinate basis $|x^\mu\rangle$. No quantum of space-time has entered yet, but quantum variables have. To take quantum space-time seriously one must eventually find a physical mechanism that freezes half the variables by self-organization (6.3).

Now we warp to full simplicity. This calls for more regulators. We follow the D line and adjoin 6 Lorentz generators $L^\mu_\nu = -L^\mu_\nu$ to the present generators x^μ, ∂_μ, assuming a fixed background Minkowski metric † that interchanges vectors and dual vectors, raising and lowering indices. This expands the 9-dimensional canonical Lie algebra dH(4) to a still singular 15-dimensional Lie algebra Lie(x^μ, ∂_μ, L^ν_μ, 1) with the commutator $AB - BA$ as Lie product $[A, B]$ and with standard commutation relations (12) for these operators. This algebra can be stabilized by warping it to a 15-dimensional orthogonal algebra dSO(6) of signature to be determined.

This simple space-time is more quantum than the Snyder space-time, which is not simple.

Notation: We label warpings by a collective argument $\mathbf{q} = (\mathbf{h}, \mathbf{c})$ with $\mathbf{h} = \{\delta q_i\} = \delta\mathbf{q}$ consisting of quantum constants like $\hbar$ and $1/c$, and with $\mathbf{c} = \{c_n\}$ consisting of quantum numbers defining values of all central invariants (see Section 3). The passage to a singular limit we write as $\mathbf{q} \to \mathbf{q}_0$. We absorb factors of i to make the variables q_i anti-Hermitian for convenience. We may omit the circumflex that indicates warping when it is redundant. The old indices $\mu, \nu = 0, 1, 2, 3$ label space-time or momentum-energy axes in the singular theory. Special constant index values X, Y label real and imaginary units in the complex plane of the singular limit. They distinguish space-time variables $L_{\mu\mathrm{X}}$ from momentum-energy variables $L_{\mu\mathrm{Y}}$ in the regular theory. Extended indices $\alpha, \beta = 0, 1, 2, 3, X, Y$ label axes in the orthogonal space that supports the regular group SO(5, 1). We may set $\hbar = c = 1$ since we hold them constant as $\mathbf{q} \to \mathbf{q}_0$. $\mathcal{X}(\mathbf{q}_0)$ is the singular quantum space and the associative algebra defined by the usual infinite-dimensional unitary representation of its Lie algebra

$$L_{\mathcal{X}}(\mathbf{q}_0) := \mathrm{Lie}(x^\mu, p_\mu, L^\nu_\mu, i) \tag{11}$$

on the function space $L^2(x^\mu)$. $L_{\mathcal{X}}(\mathbf{q}_0)$ has the familiar singular structure

$$\begin{aligned} [x^\nu, x^\mu] = 0,\ [x^\nu, p_\mu] = i\delta^\nu_\mu, \quad [x^\nu, L_{\mu\lambda}] = \delta^\nu_\mu x_\lambda - \delta^\nu_\lambda x_\mu, \quad [x^\mu, i] = 0,\\ [p^\nu, p^\mu] = 0, \quad [p^\nu, L_{\mu\lambda}] = \delta^\nu_\mu p_\lambda - \delta^\nu_\lambda p_\mu, \quad [p^\mu, i] = 0,\\ [L^{\nu\mu}, L_{\lambda\kappa}] = \delta\begin{bmatrix} \nu\ L^\mu \\ \lambda\ \kappa \end{bmatrix}, \quad [L_{\nu\mu}, i] = 0 \end{aligned} \tag{12}$$

We warp the singular Lie algebra $L_{\mathcal{X}}(\mathbf{q}_0)$ to a regular Lie algebra $\widehat{L}_{\mathcal{X}}(\mathbf{q}) \sim d\mathrm{SO}(5,1)$ as follows.

First we melt down the idol i to the Lie element $\widehat{\imath} := \widehat{r}/\mathrm{Max}\,\widehat{r}$. We return to the singular theory by "polarizing" $\widehat{r}$: freezing it at its maximum eigenvalue.

Then we rescale the dimensionless infinitesimal orthogonal transformations $L_{\beta\alpha} \in d\mathrm{SO}(5,1)$ to define warped versions of the generators of $L_{\mathcal{X}}(\mathbf{q}_0)$ in $L_{\mathcal{X}}(\mathbf{q})$. The 15 variables $L_{\beta\alpha}$ require four quantum constants $\mathbf{h} = (\delta x, \delta p, \delta L, \delta r)$, but $\delta L = \hbar = 1$ for Lorentz invariance:

$$\widehat{L}_{\nu\mu} = L_{\nu\mu}, \quad \widehat{x}_\mu = \delta x\, L_{\mu \mathrm{X}}, \quad \widehat{p}_\mu = \delta p\, L_{\mu \mathrm{Y}}, \quad \widehat{r} = \delta r\, L_{\mathrm{XY}}. \tag{13}$$

The maximum eigenvalue of $-(L_{\alpha\beta})^2$ is the same for any spatial $(\alpha\beta)$ plane, a new quantum number we write as $l^2_{\mathcal{X}}$. Evidently in the singular limit we must have

$$\delta x\, \delta p = l_{\mathcal{X}}\, \delta r\, \hbar \tag{14}$$

and we might as well impose this in general.

This warping converts the compound Lie algebra $L_{\mathcal{X}}(\mathbf{q}_0)$ to a simple Lie algebra $L_{\mathcal{X}}(\mathbf{q})$ with generators $L_{\alpha\beta}$. The canonical commutation relations survive in the warped form

$$[x^\mu(h), p_\nu] = \delta^\mu_\nu \frac{\delta x \delta p}{\delta r} r. \tag{15}$$

We construct a quantum space STiME $= \widehat{\mathcal{X}} = \mathcal{X}(\mathbf{q})$ from its Lie algebra $L_{\mathcal{X}}$ by specifying an irreducible matrix represention $R(\mathbf{h})L_{\mathcal{X}}$, whose algebra is then the operational algebra of $\widehat{\mathcal{X}}$. The singular space-time algebra is an infinite-dimensional irreducible unitary representation $R(\mathbf{h}_0)L_{\mathcal{X}}(\mathbf{c}_0)$ supported by the function space $L^2(\mathcal{X}(\mathbf{c}_0))$. To fix on one regularized STiME we must fix on quantum constants and quantum numbers $\mathbf{q}$ of the Lie algebra $L_{\mathcal{X}}$, defining a preferred basis $\mathbf{B}$ in one irreducible representation $R(\mathbf{h})L_{\mathcal{X}}(\mathbf{c})$.

And the Lie algebra $L_{\mathcal{X}}$ is specified in turn by signatures.

The quadratic mode-vector space supporting the defining representation of $L_{\mathcal{X}}$ is a 6-dimensional space $V_{\mathcal{X}}$. We form a high-dimensional representing vector space $R(\mathbf{h})V_{\mathcal{X}}$ with collective quantum constant $\mathbf{h}$, to support the physical representation $R(\mathbf{h})L_{\mathcal{X}}$. The singular space is spanned by polynomials in the coordinates, and limits thereof.

An irreducible representation $R(\mathbf{c})\mathrm{SO}(5,1)$ is defined by eigenvalues c_n of the Casimir invariants C_n, the coefficient of z^n in the invariant characteristic polynomial

$$C(z) = \det(L - z\mathbf{1}) = \sum C_n z^n \tag{16}$$

for $L \in dR(\mathbf{h})\mathrm{SO}(5,1)$. C_n vanishes for odd n because $L \sim -L$, leaving C_2, C_4, C_6. As usual iL_{XY} has eigenvalue spectrum of the form $-l, -l+1, \ldots, l-1, l$. The extreme value l is a regulant and $2l$ is an integer. In the singular limit $l \to \infty$.

Let $\mathsf{L} = (L^i{}_j)$ be the matrix whose elements are the infinitesimal generators of $dR(\mathbf{c})\mathrm{So}(5,1)$; a matrix of matrices. Then for each $n \in \mathbb{N}$, $\mathrm{Tr}\,\mathsf{L}^n$ is another convenient invariant, whose value in the chosen representation we designate by $\Lambda^{(n)}$. In particular,

$$\Lambda^{(2)} = -(L_{\mathrm{XY}})^2 - L^{\mathrm{X}\mu} L_{\mathrm{X}\mu} - L^{\mathrm{Y}\mu} L_{\mathrm{Y}\mu} + L^{\nu}_{\mu} L^{\mu}_{\nu} \tag{17}$$

The cross-terms $-L^{\mathrm{X}\mu} L_{\mathrm{X}\mu} - L^{\mathrm{Y}\mu} L_{\mathrm{Y}\mu}$ have vanishing expectation for any eigenvector of L_{XY} by the generalized uncertainty inequality. Then

$$\Lambda^{(2)} = l^2 - (\delta x)^{-2} \langle \widehat{x}^{\mu} \widehat{x}_{\mu} \rangle - (\delta p)^{-2} \langle \widehat{p}^{\mu} \widehat{p}_{\mu} \rangle + \left\langle L^{\nu}_{\mu} L^{\mu}_{\nu} \right\rangle \approx l^2 \tag{18}$$

holds for the vacuum, as an eigenvector of extreme L_{XY}. In the correspondence domain one may drop the circumflexes.

This is a warped Klein–Gordon equation with a "mass" term that depends on the STiME cooordinates and angular momentum. Wigner taught us that the scalar fields supporting irreducible representations of the Poincaré group obey Klein–Gordon wave equations. Naturally a warped group leads to a warped wave equation.

Similarly

$$\forall n \in \mathbb{N} \quad | \quad c^{(2n)} - l^{2n} \approx 0 = c^{(2n+1)} \tag{19}$$

are polynomial conditions on $\widehat{x}^{\mu}$, $\widehat{p}_{\mu}$, and L^{μ}_{ν} with coefficients depending on $\mathbf{h}$ and l.

Raising the dimension of the group has increased the number of invariants and wave equations.

The algebra $\widehat{A} = \widehat{A}(\widehat{\mathcal{X}})$ of coordinate variables of the regular STiME quantum space $\widehat{\mathcal{X}} := \mathcal{X}(\mathbf{h})$ is the operator algebra of the vector space $R(\mathbf{c})V_{\mathcal{X}}$ that we have just constructed:

$$\widehat{A} := \mathrm{Endo}\; R(\mathbf{c})V_{\mathcal{X}}. \tag{20}$$

Each factor in $R(\mathbf{c})V_{\mathcal{X}}$ contributes angular momentum ± 1 or 0 to each generator $L_{\alpha\beta}$ of $L_{\mathcal{X}}(h)$, so the eigenvalue of $i\,R(\mathbf{c})L_{\alpha\beta}$ varies from $-l$ to l in steps of 1. Now the space-time coordinates and the energy-momenta are unified under the Lie group generated by $R(\mathbf{c})L_{\mathcal{X}}$. Each has a discrete bounded spectrum with $2L+1$ values $x = i\delta x\, m$, $p = i\delta p\, m$, for $m \in \mathbb{Z}$, $|m| \in l+1$. Both operators are elements of the STiME operator algebra $A\mathcal{X}_h :=$ Endo $R(\mathbf{c})L_{\mathcal{X}}$, which replaces $\mathrm{L}^2(\mathcal{X}_0)$.

The regular quantum point of STiME can be represented as a series of l more elementary processes or chronons, all identical, a bosonic ensemble constrained to a fixed number l of elements. The chronon is a minkowskion in this model.

Next we set up a singular scalar q/c field theory on the singular quantum space-time so that we can regularize it in Section 6.3.

6.2. Singular Field Lie Algebra

We label the singular q/c limit with a suffix ($\mathbf{q}_0$) and generic q/q case with ($\mathbf{q}$), dropping the circumflex, where $\mathbf{q}$ is a collection of quantum constants and quantum numbers to be specified.

In the c scalar theory a history f of the field is a pair $(f(\cdot), p_f(\cdot)$ of a field function $f : \mathcal{X} \to \mathbb{R}$ on space-time, and a contragredient momentum function $p_f : \mathcal{X} \to \mathbb{R}$. The space of such c histories is, aside from continuity requirements,

$$\mathcal{F} = \mathbb{R}^{\mathcal{X}} =: \mathrm{D}\mathcal{X}, \tag{21}$$

a kind of linear dual of $\mathcal{X}$. The c functional Lie algebra is commutative:

$$\begin{aligned} f(x)f(x') - f(x')f(x) &= 0, \\ p_f(x)p_f(x') - p_f(x')p_f(x) &= 0, \\ p_f(x)f(x') - f(x')p_f(x) &= 0, \\ f^\dagger + f &= 0, \\ {p_f}^\dagger + p_f &= 0. \end{aligned} \tag{22}$$

The bosonic aggregate, or the quantum field, has the formal functional Lie algebra $L_{\mathcal{F}}(\mathbf{q}_0)$ generated by the operators on $\mathrm{D}\mathcal{X}$ of multiplication by $f(\cdot)$ variational differentiation $p_f = \delta_f(x) := \delta/\delta f(x)$, and the central i, subject now to the canonical relations of a $d\mathrm{H}(\infty)$,

$$\begin{aligned} f(x)f(x') - f(x')f(x) &= 0 \\ p_f(x)p_f(x') - p_f(x')p_f(x) &= 0 \\ p_f(x)f(x') - f(x')p_f(x) + i\,\hbar\delta(x - x') &= 0 \\ f^\dagger + f &= 0 \\ {p_f}^\dagger + p_f &= 0. \end{aligned} \tag{23}$$

for all $x, x' \in \mathcal{X}$. We make both f and p_f anti-Hermitian with incorporated factors of i where necessary, for the sake of the development to come.

Because of the two-story construction the operators x^{μ} and ∂_{μ} in the space-time Lie algebra $dL_{\mathcal{X}}$ can also act on the field functional Lie algebra $dL_{\mathcal{F}}$, with obvious commutation relations.

The element $i\,\hbar$ is a complete set of central invariants of this functional Lie algebra. The canonically quantized scalar field is a bosonic aggregate of individuals whose mode-vector space is $\mathrm{L}^2(\mathcal{M})$.

This is the theory we warp to regularity next.

6.3. Regular Field Lie Algebra

The Lie algebra of bosonic statistics is unstable, compound. We warp it now to a simple, stable, and finite near-bosonic statistics.

We use the procedure Σ_L of (8). A fixed io mode-vector space V for an individual quantum I is given, and a Lie algebra L on V, with a structure tensor X. It is convenient to give X on an ι-labeled replica of V in case there are other Lie algebra structures already defined on V, as in multiple quantification.

The algebra A is the operational algebra ΣV of the quantified system determined by the algebra L and the quantum constants $\mathbf{c}$. A vacuum projection $P(\text{vac}) \in \Sigma V$ then determines the vector space $\Sigma V\; P(\text{vac})$ as a mode-vector space for the quantified system.

In the case of the singular boson quantification L is the functional Lie algebra $\text{Canon}_{+}\iota^{\dagger}V$, defined by the bosonic commutation relations on the union $V = V_{\mathrm{I}} \cup V_{\mathrm{O}}$ of the input and output mode-vectors of the system. V is a partial vector space; addition works within each term but not between them.

Then the bosonic Lie algebra of the aggregate of individuals I is the target algebra $A(\mathbf{c})$ of the irreducible unitary representation $R(\mathbf{c})$ of L with the central invariant $c = i\,\hbar$ specified.

The usual creator and annihilator of the many-quantum (or quantified) theory associated with the mode-vectors v and $v^{\dagger}$ of the one-quantum theory are left multiplication $\iota^{\dagger}v$ and differentiation $(\iota^{\dagger}v)^{\dagger} = v^{\dagger}\iota$ with respect to $\iota^{\dagger}v$.

If the $v_n \in V$ form a basis of input vectors with dual output vectors, the corresponding creation/annihilation operators $a_n := \iota^{\dagger}v_n$, $c^n := v^{n\,\dagger}\iota$ obey

$$
\begin{aligned}
c^n a_m - a_m c^n - i\,\hbar\delta^n_m &= 0,\\
c^n c^m - c^m c^n &= 0,\\
a_n a_m - a_m a_n &= 0.
\end{aligned}
\tag{24}
$$

This algebra is doubly infinite-dimensional: once because bosonic quantification turns each dimension on the one-quantum space into an infinity of dimensions in the many-quantum space, and once because the one-quantum space has an infinity of dimensions, because space-time is infinite and continuous. The space-time infinity is again bosonic, arising from the fact that space-time is a bosonic aggregate of minkowskions,

Now we warp the relations (24) to simplicity and regularity by warping both boson algebras to simple orthogonal group algebras. To reduce this to an earlier problem we first transform bosonic variables a_n, c^n to canonical anti-Hermitian variables $q_n = -q_n{}^\dagger$, $p_n = -p_n{}^\dagger$ ($n \in \mathbb{N}$) using the imaginary unit i:

$$a_n = \frac{q_n/\delta q + i p_n/\delta p}{\sqrt{2}}, \quad c_n = \frac{q_n/\delta q - i p_n/\delta p}{\sqrt{2}}, \tag{25}$$

with quantum constants δq, δp, δr for dimensional reasons. Then we introduce two extra real dimensions with indices X′, Y′ forming a real vector space $V \oplus \mathbf{2}$ with vector indices $\alpha, \beta = 0, \ldots, N-1, \ldots, \mathrm{Y}$. A symmetric metric $\dagger : (V \oplus \mathbf{2}) \rightarrow (V \oplus \mathbf{2})^{\mathrm{D}}$ defines an orthogonal Lie algebra $d\mathrm{SO}(V \oplus \mathbf{2})$ generated by $(N+2) \times (N+2)$ matrices $L_{\beta\alpha}$, anti-Hermitian with respect to $\dagger$. We represent the warped simple-bosonic creators and annihilators in the simple Lie algebra $d\mathrm{SO}(V \oplus \mathbf{2})$:

$$\widehat{q}^n := \delta q L^n_{\mathrm{X}'}, \quad \widehat{p}_n = \delta p L^{\mathrm{Y}}_n, \quad \widehat{\imath} := \delta r L^{\mathrm{Y}}_{\mathrm{X}'}, \tag{26}$$

For an alternative representation see Baugh (2004).

The space-time regulation introduced a large quantum number $l_{\mathcal{X}}$, setting the maximum of the space-time iL_{XY}, and approaching ∞ in the singular limit. This determines the dimension N of the space-time mode-vector space. Now the field algebra regulation introduces a large quantum number $l_{\mathcal{F}}$ determining the maximum eigenvalue of the field $L_{\mathrm{X}'\mathrm{Y}'}$. The warped relations include

$$[\widehat{q}^m, \widehat{p}_n] = i\delta q\delta p\big(\delta^n_m L^{\mathrm{X}'}_{\mathrm{Y}'}\big) \rightarrow i\,\hbar\delta^n_m \tag{27}$$

We infer that

$$l\,\delta q\,\delta p = \hbar \tag{28}$$

We have simplified the Lie algebra and now must simplify its representation. To construct the physical variables, which typically have many more eigenvalues, we must pass from the given low-dimensional Lie algebra to a suitable irreducible orthogonal representation of dimension large enough to pass for infinite.

In the singular theory this representation is the bosonic quantification of the underlying Lie algebra, unique up to one quantum constant $\hbar$ but infinite-dimensional. Here in the regular theory the warped space-time Lie algebra is simple, that of SO(5, 1), and its representation is finite dimensional. The representation algebra is that of dSO(5, 1) defined by values of the central invariants $\mathbf{c}$. The representation of the physical basis is defined by quantum constants $\mathbf{h}$ close to their singular values.

6.4. Singular Scalar Dynamics

The usual scalar Green's function is

$$G(x'_1, \ldots, x'_n) = \langle \text{vac} \mid \phi(x'_1), \ldots, \phi(x'_n)) \mid \text{vac}\rangle \tag{29}$$

Here x'_1 is a collection of c numbers, eigenvalues of the coordinate operators $x = (x^\mu)$, and $\phi(x'_1)$ is a creation/annihilation operator associated with the position eigenvalue x'_1.

The construct G is covariant under the unitary group of basis changes for the space F of fields $\phi(x')$. Any orthonormal frame $\{\phi_\alpha\}$ for the mode-vector space of a single boson defines a generalized Green's function

$$G_{\alpha_1,\ldots,\alpha_n} = \langle \text{vac} \mid \phi_{\alpha_1}, \ldots, \phi_{\alpha_n} \mid \text{vac}\rangle \tag{30}$$

This form can survive the warping that we carry out. The nature of the one-quantum mode-vector, however, changes discontinuously at the singular limit. For example, in c space-time the coordinates x^μ all commute, and so their eigenvalues can label the mode-vector $\phi_{x'}$. But in the warped quantum space (STiME), space-time coordinates $\widehat{x}^\mu$ do not commute and their eigenvalues cannot label a basis. Instead there are commuting variables $t = \delta t\, L_{0\text{X}}$, $p_x = \delta p\, L_{1\text{Y}}$, and L_{23}, which may be supplemented by the quantum numbers c_2, c_4, c_6 as necessary to make a complete commuting set. To recover the singular Green's function from the regular we must construct coherent states that are only approximately eigenvectors of all the $\widehat{x}^\mu$.

The vacuum mode-vector $\mid \text{vac}\rangle$ of the singular quantum theory is defined by its amplitude, which has the Lagrangian form

$$\langle \phi(\cdot) \mid \text{vac}\rangle := N \exp i \left[\int d^4x\, L(\phi(x), \partial_\mu \phi(x)) \right] =: N \exp i A. \tag{31}$$

in which $A = A[\phi(\cdot)]$ is the action integral of the exponent. This gives an amplitude for each field history $\phi(\cdot)$.

The singular dynamical theory we warp is that of a free scalar meson, with Lagrangian density

$$L(\phi(x), \partial_\mu \phi(x)) := -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + m^2 \phi(x)^2 \tag{32}$$

6.5. Regular Scalar Dynamics

The free field or many-quantum action A is constructed from the one-quantum antisymmetric operator

$$A_1 := i p^\mu p_\mu + i m^2 \tag{33}$$

by quantification. To quantify A_1, we first make explicit the mode-vectors ϕ_x and their duals $\phi_x{}^\dagger$ that enter into it. We choose an x basis only for its familiarity:

$$A_1 = N \int d^4x\, \phi_x L^{xx'} \phi_{x'}{}^\dagger, \tag{34}$$

with a singular normalizer N and a singular kernel $L^{xx'}$. Then quantification replaces the one-quantum mode-vectors ϕ_x and their duals $\phi_x{}^\dagger$ by many-body operators $\iota\phi_x$ and $\phi_x{}^\dagger\iota^\dagger$ obeying bosonic commutation relations, defining the same singular algebra as a particle in infinite-dimensional space. The result is the singular action A of (31), now written

$$A = N \int d^4x\, \iota\phi_x L^{xx'} \phi_x{}^\dagger \iota^\dagger = \iota A_1 \iota^\dagger \tag{35}$$

To warp A we need only warp A_1.

To be sure, the algebra of ι and $\iota^\dagger$ is singular and infinite dimensional. Perhaps it too can be regularized. This would modify the q set theory to allow membership loops, as in Finsler set theory. But we do not iterate ι, so it introduces no singularities, and we leave ι fixed.

To warp the action A_1 we warp each operator in it. As usual, quantization requires us to order operators that no longer commute so that their product remains antisymmetric. For economy we choose the order

$$\widehat{A}_1 = \widehat{p}^\mu \widehat{\iota}\, \widehat{p}_\mu + m^2 \widehat{\iota}. \tag{36}$$

The warped action is then

$$\widehat{A} = \widehat{\iota} \widehat{A}_1 \widehat{\iota}^\dagger \tag{37}$$

The warped creators and annihilators obey warped bosonic commutation relations, those of $d\mathrm{SO}(M)$ with cosmologically large M.

Obviously, this is finite and so is the normalization constant $\widehat{N}$ replacing the infinite constant N. The exact Lorentz invariance and the approximate medium-energy Poincaré invariance are also plausible.

This warped action seriously breaks the warped symmetry group, and more symmetric ones that are still good approximations to the singular actgion in the correspondence domain are readily available. They go beyond the scope of this paper.

7. RESULTS

We have used general quantization to convert the usual singular theory of the scalar meson to a finite theory with nearly the same algebras and symmetries in a correspondence domain. This toy taught us how to general-quantize Minkowski space-time and bosonic statistics, and how to supply a relativistic finite dynamics to go with the finite quantum kinematics.

Warping space-time quantizes momentum-energy and warping momentum-energy quantizes space-time. Doing both produces a finite unified quantum space-time-i-momentum-energy space STiME.

The principle difference between this approach and most others is that we take seriously the partition of the theory into logical levels, each with its algebra, and preserve these algebras, with arbitrarily small changes, throughout the construction. This contrasts, for example, with approaches to quantum field theory that discretize space-time, discarding the Lorentz invariance, and then take a limit. Under general quantization the system determines its own quanta and requires no ad hoc discretization.

The correspondence principle fixes some combinations of the new quantum constants, quantum numbers, and regulants, leaving the rest to experiment. No infinite renormalization is needed.

Several discrete choices have to be left to experiment. For example the simplicity principle is equally satisfied along the real, complex, and quaternionic lines of simple Lie algebras. We chose the real line mainly because it is easiest and in some sense simplest, but nature may not take the way that is easiest or simplest for us.

We give necessary conditions on the defining parameters for the finite theory to converge to the usual theory in some appropriately weak sense, but we have not shown they are sufficient. This question may be sensitive to the theory under study. We have not proven that these finite results agree well enough with the finite results of the usual singular theory where they should but it seems plausible. Approximating the regular discrete spectrum by a singular continuous one is a somewhat delicate non-uniform convergence even for the harmonic oscillator.

We suspend our study of the scalar field for now in order to apply general quantization to more basic physical theories and to the most singular groups of physics today, the gauge groups.

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